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Scaling properties of the surface of the Eden model in $d = 2, 3, 4$

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Abstract. The surface of the Eden model is investigated numerically by finite-size scaling, using a strip geometry. Three different versions are studied and it shows that the one mostly used previously exhibits strong finite-size corrections. In two dimensions it is found that the surface thickness simply grows as the square root of the width of the strip for infinitely long strips. This result, as well as the results in $d = 3$ and $d = 4$, suggest that the surface of the Eden shares some properties with the equilibrium models used to describe the roughening transition. Moreover it is found that the surface thickness scales differently with the height of the cluster for infinitely large strips.

Among the theoretical models which have been introduced to describe cluster growth and aggregation phenomena, the Eden model (Eden 1961) is one of the most simple: particles are added one after another to a growing cluster with the prescription that each new particle sticks on any point of the surface of the cluster with equal probability. This model is generally considered as a prototype reference (see, for example, the proceedings of the conference on 'Kinetics of Aggregation and Gelation', edited by F Family and D P Landau (1984, Amsterdam: North-Holland) and references therein) with many potential applications in physics, chemistry and biology. One of its trivial aspects is the compact character of the resulting cluster which has a fractal dimension equal to the dimension of space (Eden 1961, Richardson 1973, Meakin 1983). If the bulk appears quite simple, the surface is of great interest. In particular, it is useful to know if the irreversible character of the growth induces some specific effect in the scaling properties of the surface compared with equilibrium models (Jasnow 1984 and references therein). Scaling properties of the surface of the Eden model have already been investigated (Petters *et al* 1979, Meakin and Witten 1983, Plischke and Rácz 1984). The most recent study concluded in an unusual scaling of the surface thickness in two dimensions (Plischke and Rácz 1984).

In this paper we present a systematic study of the surface of the Eden model in a different geometry, particularly well adapted to finite-size scaling analysis. Moreover, we introduce three different versions of the model which differ only on short-range scale. We show that the version previously used to compute the surface thickness (Peters *et al* 1979, Plischke and Racz 1984) is the one exhibiting the largest finite-size corrections. In two dimensions we find that there exists a steady state for infinitely long strips in which the surface thickness simply grows as the square root of the width of the strip, as in equilibrium models (Jasnow 1984). Our results in $d = 3$ and $d = 4$ are not inconsistent with the behaviour of equilibrium models. Moreover we find different scalings with the height and the width of the strip. A short account of this work has been published elsewhere (Jullien and Botet 1985).

We consider the geometry already used to study diffusion-limited aggregation (Jullien *et al* 1984, Rácz and Vicsek 1983, Turban and Debierre 1984, Gelband and Strenski 1985). In two dimensions, the cluster grows on a square lattice of unit lattice parameter inside a strip of width l, with periodic boundary conditions at the edge of the strip. The generalisation in d dimensions is straightforward: the section of the strip becomes a $(d-1)$-dimensional hypercube of size l. At the beginning, we consider that all sites are occupied up to height $z=0$. Then the particles are added one after another to the cluster. We have considered three different prescriptions.

In version A, we consider all *unoccupied* sites adjacent to the surface with the same probability and we choose at random one of these sites to accept the new particle. This is the simplest version to be handled on a computer and this explains why it has been so widely studied in the past (Peters *et al* 1979, Meakin 1983, Meakin and Witten 1983, Plischke and Rácz 1984). This version has been called the 'Eden model for physicists' (Vannimenus 1984).

In version B, we consider all open bonds, i.e. all bonds joining an occupied site to an unoccupied one, with the same probability, and we choose at random one of these bonds to receive the new particle on its empty edge site. Note that this bond version of the Eden model is the one originally introduced by Eden himself (Eden 1961).

In version C, we consider all *occupied* sites of the surface with the same probability. Then we choose at random one of those sites and we add the new particle on any of the nearest-neighbour empty sites with equal probability. We would like to emphasise that this prescription is conceptually very simple: any point of the cluster has the same chance to grow, i.e. to accept a new particle on any of its neighbouring empty sites.

To better show that these prescriptions are effectively different, let us consider a simple example in two dimensions, with $l=3$. A particular configuration is given in figure 1. The new particle can reach one of the empty sites adjacent to the surface,

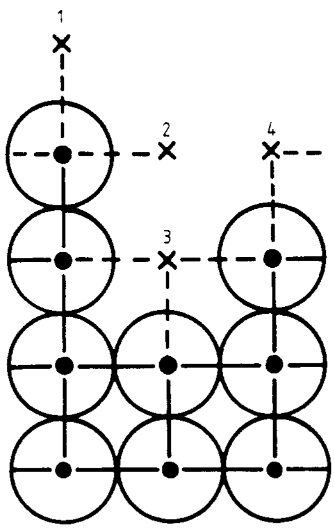

Figure 1. A typical configuration in $d=2$ with $l=3$. ● and × denote the occupied sites and the unoccupied sites adjacent to the surface, respectively. —— and - - - indicate closed bonds and open bonds, respectively.

labelled 1, 2, 3 and 4 on figure 1. The probabilities of reaching these sites (given in table 1) are different in each version. For example, site 1 which is located on the top of a tiny finger has less chance of being reached when going from model A to model C. On the other hand, site 3, which lies in a small depression, has more chance of being reached. Thus one must expect a smoother surface when going from model A to model C and this will be confirmed by the numerical results reported below. However, if one trusts the general scaling theory of critical phenomena, such small length scale differences must not affect the scaling properties for sufficiently large sizes. If the *amplitudes* may be different, the *exponents* must be the same in the three versions.

Table 1. Probability that the new particle reaches site 1, 2, 3 or 4, with model A, B or C in the case of the configuration shown in figure 1.

Site	1	2	3	4
A	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
B	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{2}{7}$
C	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{5}{8}$	$\frac{5}{24}$

We have calculated the thickness of the surface σ by the formula:

$$\sigma^2 = \sum_i (z_i - \bar{z})^2 / n_s$$

with (1)

$$\bar{z} = \sum_i z_i / n_s$$

where the sum covers the n_s surface sites labelled by index i and z_i denotes the height of site i. In model A we have also calculated σ' by

$$\sigma'^2 = \sum_{i'} (z_{i'} - \bar{z}')^2 / n'_s$$

$$\bar{z}' = \sum_{i'} z_{i'} / n'_s \qquad (1')$$

where now the sum covers the n'_s empty sites adjacent to the surface (this definition of the thickness comes naturally here). In all cases, when there are some holes, their surface is counted in the calculation of the thickness.

The surface thickness depends on two parameters, the width l of the strip and the total number of particles N which have been added from the beginning. We have conveniently replaced N by an 'effective height' h defined as

$$h = N/l$$

so that σ now depends on two independent lengths. For large l and h, general scaling arguments imply that σ takes the scaling form (Family and Vicsek 1985):

$$\sigma(l, h) \sim l^\alpha f(h/l^\gamma) \qquad (2)$$

with

$$f(x) \to \text{constant} \qquad \text{for } x \to \infty \qquad (2')$$

$$f(x) \sim x^\beta \qquad \text{for } x \to 0. \qquad (2'')$$

Since the h dependence of σ must become independent of l, for $l \gg h$, there exists the following relation between the above defined exponents:

$$\gamma = \alpha / \beta. \tag{3}$$

Consequently, if $\gamma \neq 1$, one has a different scaling for $l \ll h$, $\sigma \sim l^{\alpha}$, and for $h \ll l$, $\sigma \sim h^{\beta}$.

Equation (2′) traduces the fact that, for a given l, the system always reaches a steady state for $h \gg l$, in which $\bar{z}$ becomes equal to h, and n_s and σ saturate to constant values $n_s(l, \infty)$ and $\sigma(l, \infty)$. The existence of a steady state renders the Eden model completely different from the exactly solvable independent column model (Weeks *et al* 1976) where, for finite l, the surface thickness diverges as $h^{1/2}$ when $h \to \infty$, in all dimensions.

In two dimensions, we have done two series of calculations to determine α and β separately. The first series of calculations has been done in the steady state. n_s and σ^2 have been averaged over 1000 independent trials and also averaged in an extended portion of the steady-state regime, waiting up to $h = 40l$ for widths ranging up to $l = 192$. For the total number of sites of the surface, we have recovered the simple result

$$n_s(l, \infty) \sim l. \tag{4}$$

The estimated constant of proportionality is given in table 2. As expected this constant becomes smaller when going from model A to model C, since the surface becomes smoother. The results for $\sigma(l, \infty)$ against l are reported in figure 2 (on a log-log plot) in the three cases A, B and C. Striking differences can be seen in this figure, not only for the absolute value of σ, which decreases from model A to model C, as expected, but also for the shape of the curves. While the C curve is remarkably linear, leading to

$$\alpha = 0.50 \pm 0.03$$

the others exhibit a change of curvature which must be attributed to finite-size corrections. In case A, these effects are particularly strong: the slope decreases first, goes through a minimum $\alpha \simeq 0.37$ between $l = 24$ and $l = 48$ and then increases slowly. The differences between the three cases may also be seen directly on the typical examples shown in figure 3. In case A, the surface contains a great number of small holes which artificially increases the effective thickness. The poor convergence of model A has already been noticed by Meakin and Witten (1983).

In the other series of $2d$ calculations we have determined $\sigma(l, h)$ for h and l both tending to infinity with a constant ratio $h/l = a$. In figure 4 we report results for σ (model C) and σ' (model A) plotted against l (on a log-log plot) with $a = \frac{1}{6}$. Here also σ^2 has been averaged over 1000 independent trials. We were able to go up to

Table 2. Constant of proportionality between the number of surface sites n_s and l^{d-1}, for large l, evaluated in the steady state ($h \gg l$). The errors are of the order of 0.05.

	$d=2$			$d=3$	$d=4$
	A	B	C	C	C
n_s/l^{d-1}	2.91	1.63	1.30	1.15	1.10
n_s'/l^{d-1}	2.18				

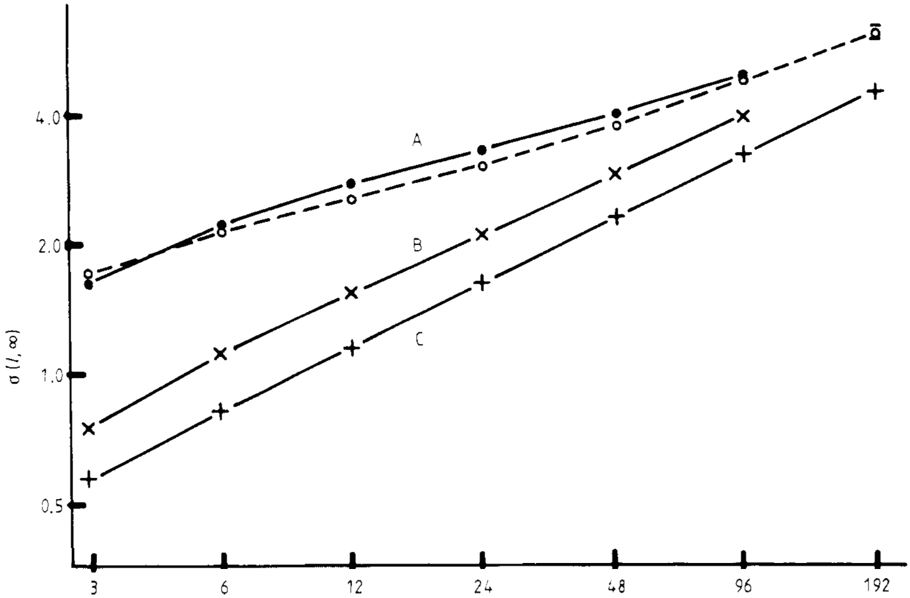

Figure 2. Numerical two-dimensional results for the l dependence of the thickness in the steady state. ●, ×, + correspond to cases A, B, C respectively. ○ corresponds to σ' (see text) in case A.

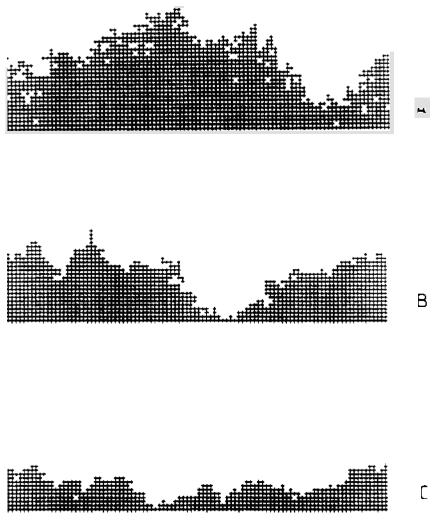

Figure 3. Typical two-dimensional examples with $l = 96$. The figure only shows the last top rows, containing surface sites.

$l = 768$, so that here the largest cluster contains $N = 98\,204$ particles. We observe the same differences between model A and C, as in figure 2. Here again the C curve is remarkably linear. However we observe that its slope is definitely different from that in figure 2. According to (2) and (2″) this implies that γ is different and greater than

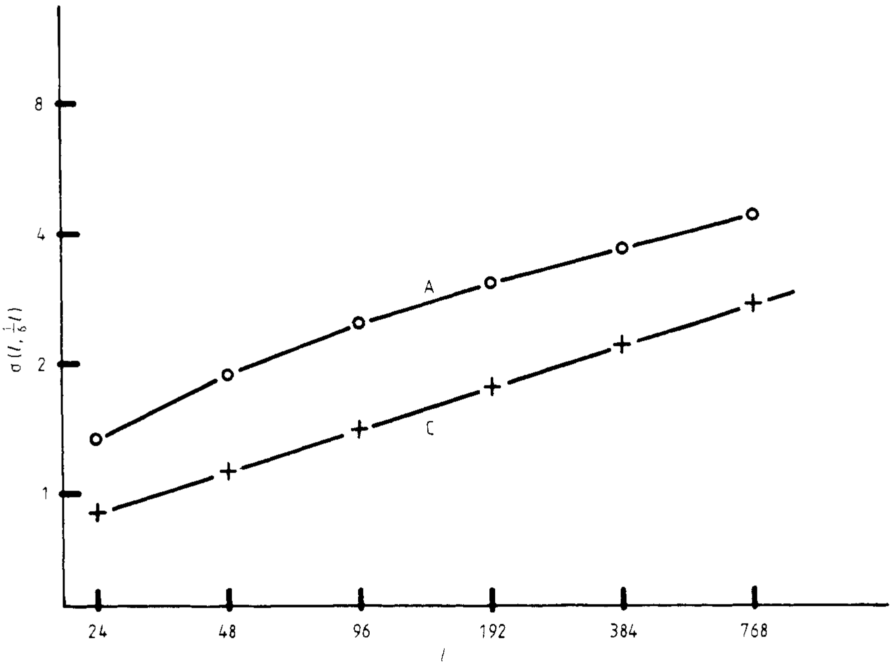

Figure 4. Numerical two-dimensional results for the l dependence of $\sigma(l, al)$ with $a = \frac{1}{6}$. ○ corresponds to σ' in case A and + corresponds to case C.

one and the slope is then directly measuring β. From the C curve we obtain:

$$\beta \simeq 0.30 \pm 0.03.$$

Combined with the preceding result for α this gives:

$$\gamma = \alpha/\beta \simeq 1.7 \pm 0.3.$$

Note that, when comparing with the spherical geometry where the cluster grows from a seed, the exponent β defined in (2″) becomes the exponent relating σ to the radius $\sigma \sim R^{\beta}$ (in the scaling regime, the spherical geometry is recovered by letting h and l tend to infinity with $h/l = \text{constant}$). Our result for β, $\beta \simeq 0.30$, must be compared with the result of Plischke and Rácz (1984): $\beta \simeq 0.37$. However these authors used version A and, as they noticed themselves, the slope of their curves was still decreasing for the largest size they could reach ($N \simeq 2500$) and their β value is certainly overestimated. Our calculations show that the C version is better adapted to calculate the exponents. With model C, we have also checked numerically the scaling form (2) by plotting $\sigma/l^{1/2}$ as a function of $h/l^{1.7}$ for different values of l (figure 5). A similar scaling form, with $\gamma \neq 1$, has recently been proposed by Family and Vicsek (1985), in a different context, and also obtained analytically by Dhar (1985) who studied an exactly solved related growth model (Edwards and Wilkinson 1982). Dhar found analytically $\alpha = \frac{1}{2}$, $\gamma = 2$. The α value is in good agreement with our numerical results while the γ value is on the verge of our error bar. More precise calculations are needed to really test numerically the equivalence between the models.

In higher dimensions, we report the results obtained with model C in the steady state only. As in $d = 2$, n_s and σ^2 have been averaged over 1000 independent trials and also averaged in an extended portion of the steady-state regime, waiting up to

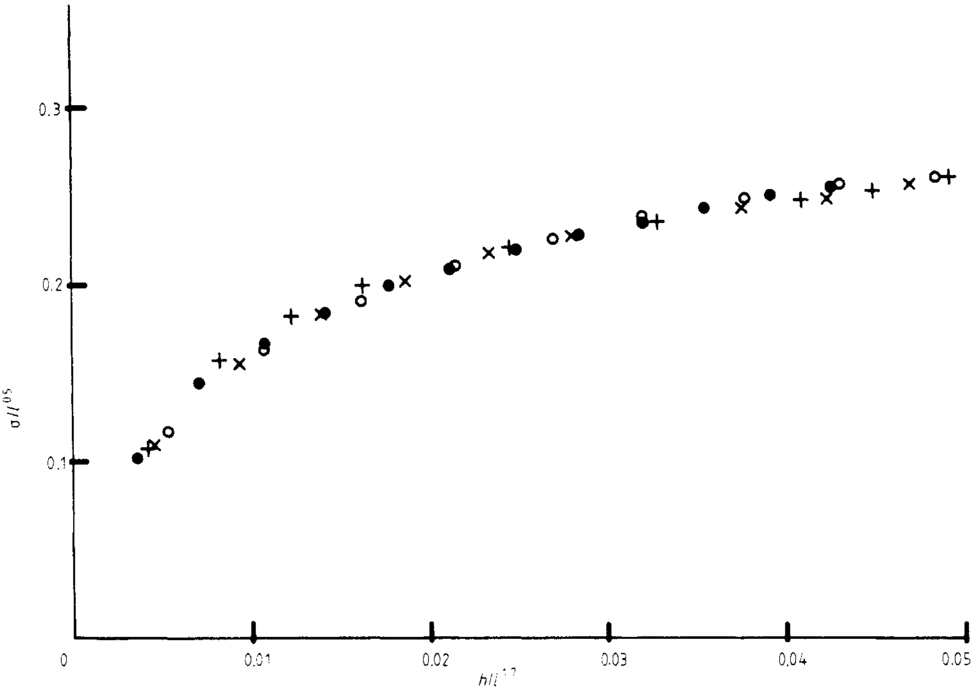

Figure 5. Scaling function in case C for $d=2$. $\sigma/l^{1/2}$ is plotted as a function of $h/l^{1.7}$ for different values of l: $+$, 48; ○, 96; ●, 192; $\times$, 384.

$h=40l$. In three and four dimensions, we were limited to $l=48$ and $l=12$, respectively. For $n_s(l,\infty)$, we have recovered the simple generalisation of the $d=2$ results:

$$n_s(l,\infty)\sim l^{d-1} \tag{5}$$

which is a trivial result (as if the surface would be completely smooth). The constant of proportionality is given in table 2 together with the $d=2$ results. The numerical results for σ against l are reported in figure 6 (on a log-log plot) and in figure 7 (on a semi-log plot). From figure 6 we could conclude that $\alpha\simeq0.20\pm0.05$ and $\alpha\simeq 0.08\pm0.06$ in $d=3$ and $d=4$, respectively. Note that such an interpretation would require larger small size corrections than in $d=2$ (which is reflected by our large error bars). On the other hand, from figure 7, we could conclude that σ would behave logarithmically in $d=3$ and would probably saturate to a constant value in $d=4$. It would be necessary to reach larger sizes to distinguish between the two possibilities. If we accept the second conclusion, the Eden model surface would behave as in several equilibrium models, such as the Ising model or models used to explain the roughening transition (Jasnow 1984, Gallavotti 1972, Francke 1980): σ would follow a square root behaviour in $d=2$, would behave logarithmically in $d=3$ and would saturate in larger dimensions. This possibility was suggested by Plischke and Rácz. This is also in agreement with the recent study by Dhar (1985).

In conclusion, our numerical results suggest that the Eden model could share some similarities with equilibrium models. However, we have found that, in the strip geometry, the surface thickness scales differently with the width (for large height) and with the height (for large width). The trivial $2d$ square root behaviour is recovered

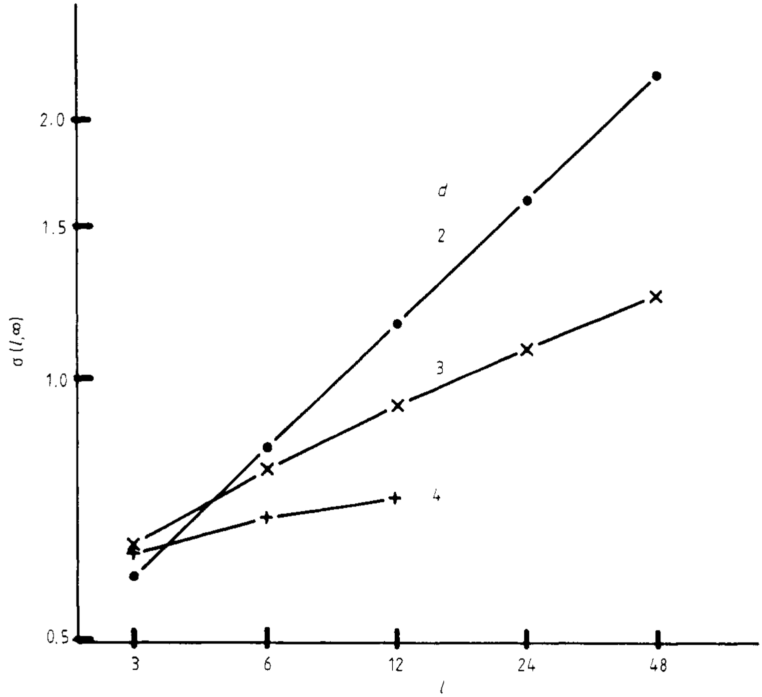

Figure 6. Numerical results for the l dependence of the thickness in the steady state in case C shown in a log-log plot. ●, ×, + correspond to $d = 2, 3, 4$, respectively.

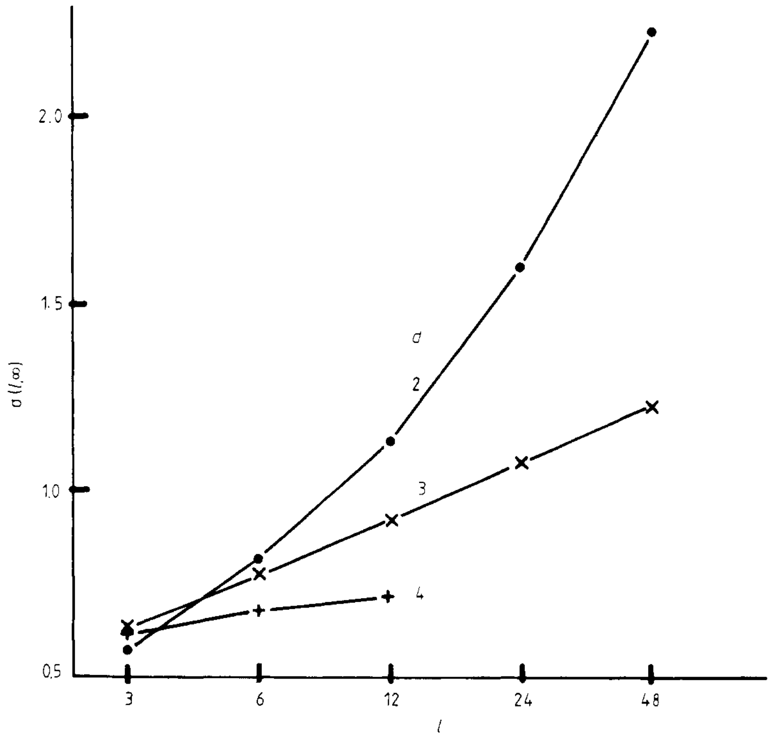

Figure 7. The same results as in figure 4 shown in a semi-log plot.

with the width (which plays the role of a size) while another type of behaviour is found with the height (which plays the role of time). This property, already found in the random deposition model (Dhar 1985), is certainly due to the irreversible kinetic character of the Eden model.

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